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Representation of Reproducing Kernels and the Lebesgue Constants on the Ball[1]

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For the weight function $(1-\|\mathbf{x}\|^2)^{\mu-1/2}$ on the unit ball, a closed formula of the reproducing kernel is modified to include the case $-1/2<\mu<0$. The new formula is used to study the orthogonal projection of the weighted L^2 space onto the space of polynomials of degree at most n, and it is proved that the uniform norm of the projection operator has the growth rate of $n^{(d-1)/2}$ for $\mu<0$, which is the smallest possible growth rate among all projections, while the rate for $\mu \geqslant 0$ is $n^{\mu+(d-1)/2}$.

Key Words: reproducing kernel; orthogonal polynomials; projection operator.

1. INTRODUCTION

For $\mathbf{x} \in \mathbb{R}^d$ let $\|\mathbf{x}\|$ denote the Euclidean norm. On the unit ball $B^d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| \leqslant 1\}$ define the weight function

$$W_\mu(\mathbf{x}) = w_\mu (1-\|\mathbf{x}\|^2)^{\mu-1/2}, \qquad w_\mu = \frac{\Gamma\left(\mu+\dfrac{d+1}{2}\right)}{\pi^{d/2}\Gamma\left(\mu+\dfrac{1}{2}\right)}$$

for $\mu>-1/2$, where the constant w_μ is chosen so that the integral of W_μ over the unit ball B^d is 1. Let Π_n^d denote the space of polynomials of degree at most n in d variables. The reproducing kernel of Π_n^d in the space $L^2(W_\mu; B^d)$ is denoted by $K_n(W_\mu; \mathbf{x}, \mathbf{y})$. For each $n \geqslant 0$, let $S_n(W_\mu)$ denote the orthogonal projection of the space $L^2(W_\mu; B^d)$ onto the subspace Π_n^d. It is the n-th partial sum of the Fourier orthogonal expansion and it can be written in terms of the reproducing kernel as

$$S_n(W_\mu; f, \mathbf{x}) = \int_{B^d} K_n(W_\mu; \mathbf{x}, \mathbf{y})\, f(\mathbf{y})\, W_\mu(\mathbf{y})\, d\mathbf{y}$$

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for $f \in L^2(W_\mu; B^d)$. The Lebesgue function of $S_n(W_\mu)$ is defined by

$$L_n(W_\mu; \mathbf{x}) = \int_{B^d} |K_n(W_\mu; \mathbf{x}, \mathbf{y})| \, W_\mu(\mathbf{y}) \, d\mathbf{y}, \tag{1.1}$$

which is in fact a function that depends on $\|\mathbf{x}\|$ only. Moreover, the constant $\|L_n(W_\mu)\|_\infty = \max_{\mathbf{x} \in B^d} |L_n(W_\mu; \mathbf{x})|$ is called the Lebesgue constant.

Recently, a closed formula of the reproducing kernel was discovered in [7, 9] for $\mu \geqslant 0$, which takes the form

(1.2)

$$K_n(W_\mu; \mathbf{x}, \mathbf{y}) = c_\mu \frac{2\Gamma\left(\mu + \frac{d+2}{2}\right) \Gamma(n+2\mu+d)}{\Gamma(2\mu+d+1)\, \Gamma\left(n+\mu+\frac{d}{2}\right)} \times \int_{-1}^{1} P_n^{(\mu+d/2,\, \mu+(d-2)/2)}\left(\langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, t\right) \times (1-t^2)^{\mu-1}\, dt,$$

where $\langle \mathbf{x}, \mathbf{y}\rangle$ denotes the usual Euclidean inner product of $\mathbf{x}, \mathbf{y} \in B^d$, $P_n^{(a,b)}$ denotes the standard Jacobi polynomial of degree n (cf. [6]), and the constant c_μ is given by

$$c_\mu = \left[\int_{-1}^{1} (1-t^2)^{\mu-1}\, dt\right]^{-1} = \frac{\Gamma(\mu+1/2)}{\sqrt{\pi}\,\Gamma(\mu)}.$$

For $\mu = 0$, the formula (1.2) holds under the limit

$$\lim_{\mu \to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1}\, dt = \tfrac{1}{2}[f(1) + f(-1)]. \tag{1.3}$$

This closed formula has been used in [9] to study the Cesàro summability of orthogonal expansions with respect to W_μ and in [10] for constructing cubature formulae. Because the integral has the weight function $(1-t^2)^{\mu-1}$, the formula does not hold for $\mu < 0$.

In [1], another expression of $K_n(W_\mu; \mathbf{x}, \mathbf{y})$ was found in the form of

$$K_n(W_\mu; \mathbf{x}, \mathbf{y}) = \frac{\sqrt{\pi}\,\Gamma(\mu+1/2)}{\Gamma(\beta+3/2)\,\Gamma(n+\beta+1)} \times \sum_{k=0}^{[n/2]} \frac{\Gamma(n+2\beta+2k+1)}{2^{4k+2\beta+1}\Gamma(k+\mu+1/2)\, k!} \times (1-\|\mathbf{x}\|^2)^k (1-\|\mathbf{y}\|^2)^k P_{n-2k}^{(\beta+1+2k,\, \beta+2k)}(\langle \mathbf{x}, \mathbf{y}\rangle), \tag{1.4}$$

in which $\beta = \mu + (d-2)/2$ (The constant in front of the sum is different from that in [1] since we use the normalized weight function). The case $\mu = 1/2$ was discovered earlier in [4].

We shall prove in the following section that (1.4) follows from (1.2) for $\mu \geqslant 0$ by a very simple argument. Evidently, the single integral is easier to work with than the sum in (1.4). In [1] it was observed, based on the formula (1.4), that

$$L_n(W_\mu; \mathbf{x}) \sim n^{\mu + \frac{d-1}{2}}, \qquad \|\mathbf{x}\| = 1$$

for all $\mu > -1/2$ (throughout this paper, the notation $A \sim B$ means that there are two constants c_1 and c_2 such that $c_1 \leqslant A/B \leqslant c_2$); and it was pointed out that the use of (1.4) for estimating the Lebesgue function for $\|\mathbf{x}\| < 1$ is hardly possible. On the other hand, the formula (1.2) can be used to prove the following result.

THEOREM 1.1. *For* $\mu \geqslant 0$, $\|L_n(W_\mu)\|_\infty \sim n^{\mu + \frac{d-1}{2}}$.

In fact, the proof of this theorem follows exactly as in [9] in which the Cesàro summability is studied. Moreover, the proof shows that the maximum of $\|L_n(W_\mu)\|_\infty$ is obtained on the boundary of B^d. It was also observed in [1] that $L_n(W_\mu; 0) \sim n^{(d-1)/2}$ for all $\mu > -1/2$ and it was conjectured accordingly that $\|L_n(W_\mu)\|_\infty \sim n^{(d-1)/2}$ for $\mu < 0$. We shall derive an integral formula for the case of $\mu < 0$ through integration by parts and analytic continuation. The new formula is then used to prove the following result.

THEOREM 1.2. *For* $-1/2 < \mu < 0$ *and* $d \geqslant 3$, $\|L_n(W_\mu)\|_\infty \sim n^{\frac{d-1}{2}}$.

This result is interesting in light of the general result in [5], which states that every projection L of the space of continuous functions onto Π_n^d satisfies $\|L\|_\infty \geqslant c n^{(d-1)/2}$ for $d \geqslant 2$, where c is a constant depending only on d. Hence, Theorem 1.2 shows that $\|L_n(W_\mu)\|_\infty$ has the smallest possible rate of growth for $\mu < 0$. We believe that Theorem 1.2 should hold for $d = 2$ as well. It is worth to point out that $\|L_n(W_0)\|$ is related to the Lebesgue constant on the sphere S^d, since the orthogonal polynomials with respect to W_0 are related to the spherical harmonics on S^d; see for example [3, Chap. 12] and [2, Chap. 3].

2. REPRESENTATION OF REPRODUCING KERNELS

For $n \geqslant 0$ and $\alpha \in \mathbb{N}_0^d$, let $\{P_\alpha^n\}_{|\alpha| = n}$ be a basis of orthonormal polynomials of degree n with respect to the weight function W_μ. The reproducing kernel $K_n(W_\mu)$ can be written as

$$(2.1) \quad K_n(W_\mu; \mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} P_k(W_\mu; \mathbf{x}, \mathbf{y}), \qquad P_k(W_\mu; \mathbf{x}, \mathbf{y}) = \sum_{|\alpha| = k} P_\alpha^k(\mathbf{x}) P_\alpha^k(\mathbf{y}),$$

where $P_k(W_\mu)$ is the reproducing kernel of the subspace of orthogonal polynomials of degree k. For $\mu \geqslant 0$, we have ([7, 9])

$$P_k(W_\mu; \mathbf{x}, \mathbf{y}) = c_\mu \frac{k+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}} \times \int_{-1}^{1} C_k^{\mu+\frac{d-1}{2}}(\langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, t) \times (1-t^2)^{\mu-1}\, dt, \tag{2.2}$$

where $\mathbf{x}, \mathbf{y} \in B^d$, $C_k^\lambda(t)$ is the Gegenbauer polynomial orthogonal with respect to the weight function

$$w_\lambda(t) = \frac{\Gamma(\lambda+1)}{\sqrt{\pi}\,\Gamma(\lambda+1/2)} (1-t^2)^{\lambda-1/2}, \qquad \lambda \geqslant 0,$$

which is normalized to have unit integral on $[-1, 1]$. It is known that the reproducing kernel $K_n(w_\lambda)$ of the polynomials of degree n in $L^2(w_\lambda; [-1, 1])$ satisfies

$$K_n(w_\lambda; 1, t) = \sum_{k=0}^{n} \frac{k+\lambda}{\lambda} C_k^\lambda(t) = \frac{\Gamma(\lambda+1/2)\,\Gamma(n+2\lambda+1)}{\Gamma(2\lambda+1)\,\Gamma(n+\lambda+1/2)} P_n^{(\lambda+1/2,\, \lambda-1/2)}(t) \tag{2.3}$$

([6, (4.5.3), p. 71], the constant is different since we use the normalized weight function). The formula (1.2) follows from (2.2) and the second equality of (2.3). Moreover, using the first equation of (2.3), it follows from (2.2) that

$$K_n(W_\mu; \mathbf{x}, \mathbf{y}) = c_\mu \int_{-1}^{1} K_n(w_{\mu+\frac{d-1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, t)(1-t^2)^{\mu-1}\, dt.$$

It is this formula that has been used for studying the summability of the orthogonal expansions with respect to W_μ.

We derive an integral formula of $K_n(W_\mu; \mathbf{x}, \mathbf{y})$ for $\mu < 0$. For this purpose, let

$$f^{\pm}_{\mu,n}(t, \mathbf{x}, \mathbf{y}) = \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}} \left[C_n^{\mu+\frac{d-1}{2}}\left(\langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, t\right)\right.$$
$$\left.\pm C_n^{\mu+\frac{d-1}{2}}\left(\langle \mathbf{x}, \mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, t\right)\right].$$

Changing variable $t \to -t$ on $[-1, 0]$, we can write $P_n(W_\mu)$ as

$$P_n(W_\mu; \mathbf{x}, \mathbf{y}) = c_\mu \int_0^1 f^{+}_{\mu,n}(t, \mathbf{x}, \mathbf{y})(1-t^2)^{\mu-1}\, dt.$$

Therefore, for $\mu > 0$, using integration by parts and the fact that $c_\mu/\mu = \Gamma(\mu+1/2)/(\sqrt{\pi}\,\Gamma(\mu+1))$ we obtain

$$P_n(W_\mu; \mathbf{x}, \mathbf{y}) = \frac{\Gamma(\mu+1/2)}{\sqrt{\pi}\,\Gamma(\mu+1)} \left[f^{+}_{\mu,n}(0, \mathbf{x}, \mathbf{y}) + \int_0^1 \frac{d}{dt}\left(f^{+}_{\mu,n}(t, \mathbf{x}, \mathbf{y})(1+t)^{\mu-1}\right) \times (1-t)^\mu\, dt \right].$$

Since the Gamma function $\Gamma(z)$ is analytic if $\Re z > 0$, the right hand side of this formula makes sense for $-1/2 < \mu < 0$ as well. Analytic continuation shows that the above formula holds for $\mu > -1/2$. Hence, using the formula

$$\frac{d}{dt} C_n^\lambda(t) = 2\lambda C_{n-1}^{\lambda+1}(t)$$

([6, (4.7.14), p. 81]), we obtain that for $\mu > -1/2$,

$$P_n(W_\mu; \mathbf{x}, \mathbf{y}) = \frac{\Gamma(\mu+1/2)}{\sqrt{\pi}\,\Gamma(\mu+1)} \left[2\, \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}}\, C_n^{\mu+\frac{d-1}{2}}(\langle \mathbf{x}, \mathbf{y}\rangle) \right.$$
$$+2\left(\mu+\frac{d+1}{2}\right)\sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}$$
$$\times \int_0^1 f^{-}_{\mu+1,n-1}(t, \mathbf{x}, \mathbf{y})(1+t)^{\mu-1}(1-t)^\mu\, dt$$
$$\left. +(\mu-1)\int_0^1 f^{+}_{\mu,n}(t, \mathbf{x}, \mathbf{y})(1+t)^{\mu-2}(1-t)^\mu\, dt \right].$$

In particular, using (2.1) and (2.3), we get the desired formula of the reproducing kernel, which we summarize as a theorem.

THEOREM 2.1. *For* $\mu > -1/2$,

$$K_n(W_\mu; \mathbf{x}, \mathbf{y}) = \frac{\Gamma(\mu+1/2)}{\sqrt{\pi}\,\Gamma(\mu+1)} \Big\{ 2K_n(w_{\mu+\frac{d-1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle) + 2\left(\mu+\frac{d+1}{2}\right)$$
$$\times \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2} \int_0^1 [K_{n-1}(w_{\mu+\frac{d+1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle$$
$$+\sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, t) - K_{n-1}(w_{\mu+\frac{d+1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle$$
$$-\sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, t)](1+t)^{\mu-1}\,(1-t)^\mu\, dt$$
$$+(\mu-1)\int_0^1 [K_n(w_{\mu+\frac{d-1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, t)$$
$$+K_n(w_{\mu+\frac{d-1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, t)]$$
$$\times (1+t)^{\mu-2}\,(1-t)^\mu\, dt \Big\}.$$

Complicated as it is, the formula will help us to estimate the Lebesgue constant in the following section.

Next we show that (1.4) follows from (1.2) for $\mu \geqslant 0$. The proof uses the following lemma.

LEMMA 2.2. *Let* g *be a polynomial of degree* n *in one variable and let* $\mu > 0$. *Then*

$$\int_{-1}^1 g(u+vt)(1-t^2)^{\mu-1}\, dt = \sum_{k=0}^{[n/2]} \frac{\Gamma(k+1/2)\,\Gamma(\mu)}{\Gamma(\mu+k+1/2)}\, g^{(2k)}(u)\, \frac{v^{2k}}{(2k)!}.$$

Proof. The stated formula follows from the Taylor expansion of $g(u+vt)$ at the point $t=0$ and the beta integral. ∎

Proof of (1.4). Let $\beta = \mu + (d-2)/2$. Using Lemma 2.2 with $g(t) = P_n^{(\beta+1,\beta)}(t)$, $u = \langle \mathbf{x}, \mathbf{y}\rangle$, $v = \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}$, we get

$$K_n(W_\mu; \mathbf{x}, \mathbf{y}) = c_\mu \frac{2\Gamma\left(\mu+\frac{d+2}{2}\right)\Gamma(n+2\mu+d)}{\Gamma(2\mu+d+1)\,\Gamma\left(n+\mu+\frac{d}{2}\right)}$$
$$\times \sum_{k=0}^{[n/2]} \frac{\Gamma(k+1/2)\,\Gamma(\mu)}{\Gamma(\mu+k+1/2)(2k)!}\, \frac{d^{2k} P_n^{(\beta+1,\beta)}}{dt^{2k}}(\langle \mathbf{x}, \mathbf{y}\rangle)$$
$$(1-\|\mathbf{x}\|^2)^k\,(1-\|\mathbf{y}\|)^k.$$

Using the formula (see [6, (4.21.7), p. 63])

$$\frac{d^{2k}}{dt^{2k}} P_n^{(\beta+1, \beta)}(t) = \frac{(n+2\beta+2)_{2k}}{2^{2k}} P_{n-2k}^{(\beta+2k+1, \beta+2k)}(t)$$

we end up with (1.4) for $\mu > 0$. The formula also holds under the limit (1.3) for $\mu = 0$. ▌

Let us point out that an analogue formula derived in [1] for the Hermite weight function $W(\mathbf{x}) = \pi^{-d/2} e^{-\|\mathbf{x}\|^2}$ on $\mathbb{R}^d$ can be derived likewise. Let $K_n(\mathbf{x}, \mathbf{y})$ denote the reproducing kernel of Π_n^d in the space $L^2(W; \mathbb{R}^d)$. Then the following formula is proved in [1].

THEOREM 2.3. *For $n \geqslant 0$,*

$$K_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} \frac{2^k}{k!} \langle \mathbf{x}, \mathbf{y} \rangle^k L_{[(n-k)/2]}^{k+d/2}(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2), \tag{2.4}$$

where L_n^α denotes the Laguerre polynomial of index α defined by

$$\sum_{n=0}^{\infty} L_n^\alpha(t)\, z^n = (1-z)^{-\alpha-1}\, e^{-tz/(1-z)}, \qquad |z| < 1. \tag{2.5}$$

Remark 2.4. We note that the formula (2.4) is stated in [1] as summation over $0 \leqslant k \leqslant [n/2]$, a close look at the proof in [1] shows that $[n/2]$ should be n; see also the proof below. As a quick check, note that if $0 \leqslant k \leqslant [n/2]$, then the lowest degree of L_k^α appears in (2.4) would be $k = [(n-[n/2])/2]$, comparing with the definition of K_m^σ at the bottom of p. 454 of [1], which contains terms with $k = 0, 1, \ldots$.

A family of orthogonal polynomials with respect to $W(\mathbf{x})$ is given by the product Hermite polynomials $H_\alpha(\mathbf{x}) = H_{\alpha_1}(x_1) \cdots H_{\alpha_d}(x_d)$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. The reproducing kernel $P_n(\mathbf{x}, \mathbf{y})$ of the orthogonal polynomials of degree n is given by

$$P_n(\mathbf{x}, \mathbf{y}) = 2^{-n} \sum_{|\alpha|=n} H_\alpha(\mathbf{x})\, H_\alpha(\mathbf{y}) / (\alpha_1! \cdots \alpha_d!),$$

where $|\alpha| = \alpha_1 + \cdots \alpha_d$, and the classical Mehler formula for the Hermite polynomials states that ([3, vol. 2, p. 194])

$$\sum_{n=0}^{\infty} P_n(\mathbf{x}, \mathbf{y})\, z^n = \frac{1}{(1-z^2)^{\frac{d}{2}}} \exp\left[\frac{-z^2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) + 2z\langle \mathbf{x}, \mathbf{y} \rangle}{1-z^2}\right],$$

which implies, upon multiplying by $(1-z)^{-1}=\sum_{k=0}^{\infty} z^k$, that

$$\sum_{n=0}^{\infty} K_n(\mathbf{x},\mathbf{y})\, z^n = \frac{1+z}{(1-z^2)^{\frac{d+2}{2}}} \exp\left[\frac{-z^2(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2)+2z\langle \mathbf{x},\mathbf{y}\rangle}{1-z^2}\right]. \tag{2.6}$$

Using (2.5) and the Taylor expansion, we obtain that the right hand side of the above equation is given by

$$\begin{aligned} RHS &= (1+z)\sum_{n=0}^{\infty} L_n^{d/2}(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2-2\langle \mathbf{x},\mathbf{y}\rangle/z)\, z^{2n} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} L_{n-k}^{k+d/2}(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2)\, \frac{(2\langle \mathbf{x},\mathbf{y}\rangle)^k}{k!}\,(z^{2n-k}+z^{2n+1-k}), \end{aligned}$$

where we have used the fact that $\frac{d}{dt} L_n^a(t) = -L_{n-1}^{a+1}(t)$ ([6, (5.1.14), p. 102]). Changing the order of summation with $2n-k=m$ and $2n-k+1=m$, respectively, which implies that $n-k=(m-k)/2\geqslant 0$ and $n-k=(m-k-1)/2\geqslant 0$, that is, $m\equiv k(2)$ and $m-1\equiv k(2)$, respectively ($m\equiv k(2)$ means $m-k\equiv 0 \bmod 2$), we conclude that

$$\begin{aligned} RHS &= \sum_{m=0}^{\infty}\Bigg(\sum_{\substack{k=0\\ m\equiv k(2)}}^{m} \frac{2^k}{k!}\langle \mathbf{x},\mathbf{y}\rangle^k L_{(m-k)/2}^{k+d/2}(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2) \\ &\qquad + \sum_{\substack{k=0\\ m-1\equiv k(2)}}^{m-1} \frac{2^k}{k!}\langle \mathbf{x},\mathbf{y}\rangle^k L_{(m-k-1)/2}^{k+d/2}(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2)\Bigg) z^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{k=0}^{m} \frac{2^k}{k!}\langle \mathbf{x},\mathbf{y}\rangle^k = L_{[(m-k)/2]}^{k+d/2}(\|\mathbf{x}\|^2+\|\mathbf{y}\|^2)\right) z^m. \end{aligned}$$

A comparison with to the left hand side of (2.6) gives the desired formula (2.4).

3. NORM OF THE PROJECTION OPERATOR

For $\mu\geqslant 0$, the estimate of the Lebesgue constant reduces to a problem of one variable. Indeed, we have

THEOREM 3.1. *Let $\lambda=\mu+(d-1)/2$. For $\mu\geqslant 0$ and $\|\mathbf{e}\|=1$,*

$$\begin{aligned} L_n(W_\mu;\mathbf{x}) &:= \int_{B^d} |K_n(W_\mu;\mathbf{x},\mathbf{y})|\, W_\mu(\mathbf{y})\, d\mathbf{y} \\ &\leqslant \int_{B^d} |K_n(W_\mu;\mathbf{e},\mathbf{y})|\, W_\mu(\mathbf{y})\, d\mathbf{y} \\ &= \int_{-1}^{1} |K_n(w_\lambda;1,t)|\, w_\lambda(t)\, dt. \end{aligned}$$

The proof of this theorem follows exactly the proof of Theorem 5.3 (the inequality) and the proof of Theorem 5.2 (the equal sign) of [9] by taking $\delta = 0$ there. There is an easier and much more general proof in the framework of weight functions invariant under a reflection group, which essentially comes down to an integration formula ([8]) for the intertwining operator in Dunkl's theory of h-harmonics; see [2]. Theorem 1.1 follows from the above theorem upon using (2.3) and the estimate of the integral of $|P_n^{(\alpha,\beta)}|$ in [6, (7.34.1), p. 173].

For the case of $\mu < 0$, however, the above theorem no longer holds. In fact, we have

$$\int_{-1}^{1} |K_n(w_\lambda; 1, t)|\, w_\lambda(t)\, dt \sim n^{\mu + \frac{d-1}{2}},$$

which grows slower than that of $n^{\frac{d-1}{2}}$ if $\mu < 0$. In order to deal with the case of $\mu < 0$, we use the formula of $K_n(W_\mu)$ in Theorem 2.1 and the following estimate of the Jacobi polynomials ([6, (7.32.5) and (4.1.3)]).

LEMMA 3.2. *For $\alpha > -1$ and $t \in [0, 1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \leqslant cn^{-1/2}(1 - t + n^{-2})^{-(\alpha+1/2)/2}.$$

A similar estimate on $[-1, 0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^n = P_n^{(\beta,\alpha)}(-t)$. We will also use the following formula that holds for $f\colon \mathbb{R} \mapsto \mathbb{R}$,

$$\text{(3.1)} \qquad \int_{S^{d-1}} f(\langle \mathbf{x}, \mathbf{y} \rangle)\, d\omega(\mathbf{y}) = \sigma_{d-2} \int_{-1}^{1} f(s\, \|\mathbf{x}\|)(1 - s^2)^{\frac{d-3}{2}}\, ds, \qquad \mathbf{x} \in \mathbb{R}^d,$$

where σ_{d-2} is the surface area of S^{d-2}. The proof of Theorem 1.2 is divided into three parts, corresponding to the three terms in the formula of $K_n(W_\mu)$ in Theorem 2.1. In the rest of the paper we use c to denote a generic constant, which depends only on d and μ and whose value may be different from line to line.

PROPOSITION 3.3. *For $-1/2 < \mu < 0$ and $\mathbf{x} \in B^d$,*

$$J_1 := \int_{B^d} |K_n(w_{\mu + \frac{d-1}{2}}; 1, \langle \mathbf{x}, \mathbf{y} \rangle)|\, W_\mu(\mathbf{y})\, d\mathbf{y} \leqslant c n^{\mu + \frac{d-1}{2}}.$$

Proof. Using the formula (3.1) and the polar coordinates,

$$\begin{aligned} J_1 &= c \int_0^1 r^{d-1} \int_{S^{d-1}} |K_n(w_{\mu + \frac{d-1}{2}}; 1, r\langle \mathbf{x}, \mathbf{y}' \rangle)|\, d\omega(\mathbf{y}')(1 - r^2)^{\mu - \frac{1}{2}}\, dr \\ &= c \int_0^1 r^{d-1} \int_{-1}^{1} |K_n(w_{\mu + \frac{d-1}{2}}; 1, r\, \|\mathbf{x}\|\, s)|\, (1 - s^2)^{\frac{d-3}{2}}\, ds (1 - r^2)^{\mu - \frac{1}{2}}\, dr. \end{aligned}$$

Changing variable $rs = t$ in the inner integral and then the order of integration, we get

$$J_1 = c \int_{-1}^{1} |K_n(w_{\mu+\frac{d-1}{2}}; 1, t\,\|\mathbf{x}\|)| \int_{|t|}^{1} r(t^2 - r^2)^{\frac{d-3}{2}} (1-r^2)^{\mu-\frac{1}{2}}\, dr\, dt$$

$$= c \int_{-1}^{1} |K_n(w_{\mu+\frac{d-1}{2}}; 1, t = \|\mathbf{x}\|)| (1-t^2)^{\mu+\frac{d-2}{2}}\, dt,$$

where the second equation follows from the beta integral. Hence, using (2.3) and $\Gamma(n+2\lambda+1)/\Gamma(n+\lambda+1/2) \sim n^{\lambda+1/2}$ as well as Lemma 3.2, we conclude that

$$J_1 \leqslant c n^{\mu+\frac{d}{2}} \int_{-1}^{1} |P_n^{(\mu+\frac{d}{2}, \mu+\frac{d-2}{2})}(t\,\|\mathbf{x}\|)| (1-t^2)^{\mu+\frac{d-2}{2}}\, dt$$

$$\leqslant c n^{\mu+\frac{d-1}{2}} \int_0^1 \frac{(1-t)^{\mu+\frac{d-2}{2}}\, dt}{(1-t\,\|\mathbf{x}\| + n^{-2})^{(\mu+\frac{d+1}{2})/2}},$$

where we have used the fact that the estimate over the interval $[-1, 0]$ is smaller than the one over $[0, 1]$, upon changing variable $t \to -t$, since $\alpha = \mu + d/2 > \mu + (d-2)/2 = \beta$. The trivial inequality $1 - t\,\|\mathbf{x}\| \geqslant 1 - t$ and the fact that $\mu + (d-2)/2 - (\mu + (d+1)/2)/2 = (\mu + (d-1)/2)/2 - 1 > -1$ for $d \geqslant 2$ show that the last integral is bounded by a constant. ▮

For the second term in the formula of $K_n(W_\mu)$ in Theorem 2.1, we only need to deal with the term

$$A_n(\mathbf{x}, \mathbf{y}) := \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2} \times \int_0^1 K_{n-1}(w_{\mu+\frac{d+1}{2}}; 1, \langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, s) \times (1+s)^{\mu-1} (1-s)^{\mu}\, ds.$$

The other term with $\langle \mathbf{x}, \mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, s$ is similar. First we derive an upper bound of $A_n(\mathbf{x}, \mathbf{y})$.

LEMMA 3.4. *For $-1/2 < \mu < 0$,*

$$|A_n(\mathbf{x}, \mathbf{y})| \leqslant c\, n^{\mu+\frac{d}{2}} |P_n^{(\mu+\frac{d}{2}, \mu+\frac{d-2}{2})}(\langle \mathbf{x}, \mathbf{y}\rangle)| + c n^{\frac{d-1}{2}} \frac{1}{(1-\langle \mathbf{x}, \mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2} + n^{-2})^{(\mu+\frac{d+1}{2})/2}} + c n^{\frac{d-1}{2}} \frac{\sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}}{(1-\langle \mathbf{x}, \mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2} + n^{-2})^{(\mu+\frac{d+3}{2})/2}}.$$

Proof. Using (2.3) we get

$$\begin{aligned}A_n(\mathbf{x},\mathbf{y}) &= \mathcal{O}(1)\, n^{\mu+\frac{d+2}{2}} \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2} \\ &\quad\times \int_0^1 P_{n-1}^{(\mu+\frac{d+2}{2},\,\mu+\frac{d}{2})}(\langle \mathbf{x},\mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, s) \\ &\quad\times (1+s)^{\mu-1}\,(1-s)^{\mu}\, ds \\ &= \mathcal{O}(1)\, n^{\mu+\frac{d+2}{2}} \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\left[\int_0^{1-n^{-1}} + \int_{1-n^{-1}}^1 \cdots\right].\end{aligned}$$

For the integral over $[0, 1-n^{-1}]$, we use the formula

$$\frac{d}{dt} P_n^{(\alpha,\beta)}(t) = \frac{1}{2}(n+\alpha+\beta+1)\, P_{n-1}^{(\alpha+1,\beta+1)}(t)$$

and integrate by parts once to get

$$\begin{aligned}A_{n,1}(\mathbf{x},\mathbf{y}) &:= \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\int_0^{1-n^{-1}} \cdots \\ &= 2(n+2\mu+d)^{-1} \times \Big[\, n^{-\mu}(2-n)^{\mu-1} \\ &\quad P_n^{(\mu+\frac{d}{2},\,\mu+\frac{d-2}{2})}(\langle \mathbf{x},\mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\,(1-n^{-1})) \\ &\quad - P_n^{(\mu+\frac{d}{2},\,\mu+\frac{d-2}{2})}(\langle \mathbf{x},\mathbf{y}\rangle) \\ &\quad + \int_0^{1-n^{-1}} P_n^{(\mu+\frac{d}{2},\,\mu+\frac{d-2}{2})}(\langle \mathbf{x},\mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}\, s) \\ &\quad \times \frac{d}{ds}\{(1+s)^{\mu-1}\,(1-s)^{\mu}\}\, ds\Big].\end{aligned}$$

Using Lemma 3.2 and replacing s in the denominator by 1, which makes the rational expression larger, we conclude that the third term in $A_{n,1}$ is bounded by

$$\begin{aligned}&cn^{-3/2}\int_0^{1-n^{-1}} \frac{1}{\left(\begin{array}{l}(1-\langle \mathbf{x},\mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2} \\ \quad \times \sqrt{1-\|\mathbf{y}\|^2}\, s + n^{-2})^{(\mu+\frac{d+1}{2})/2}\end{array}\right)}(1-s)^{\mu-1}\, ds \\ &\leqslant cn^{-\mu-3/2}\frac{1}{(1-\langle \mathbf{x},\mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2} + n^{-2})^{(\mu+\frac{d+1}{2})/2}},\end{aligned}$$

since $\mu < 0$. The lemma also shows that the first term is bounded by the same upper bound. Hence, we conclude that

$$|A_{n,1}(\mathbf{x},\mathbf{y})| \leqslant cn^{-1}\,|P_n^{(\mu+\frac{d}{2},\,\mu+\frac{d-2}{2})}(\langle \mathbf{x},\mathbf{y}\rangle)| \\ + cn^{-\mu-3/2}\frac{1}{(1-\langle \mathbf{x},\mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}+n^{-2})^{(\mu+\frac{d+1}{2})/2}}.$$

Hence, $\mathcal{O}(1)\,n^{\mu+\frac{d+2}{2}}\,|A_{n,1}(x,y)|$ is bounded by the first two terms of the stated estimate of $|A_n(\mathbf{x},\mathbf{y})|$.

Next, we use Lemma 3.2 and the fact that $\mu<0$ to get that

$$\left|\int_{1-n^{-1}}^{1} P_{n-1}^{(\mu+\frac{d+2}{2},\,\mu+\frac{d}{2})}(\langle \mathbf{x},\mathbf{y}\rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\,s)(1+s)^{\mu-1}\,(1-s)^{\mu}\,ds\right| \\ \leqslant cn^{-1/2}\int_{1-n^{-1}}^{1}\frac{1}{(1-\langle \mathbf{x},\mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\,s+n^{-2})^{(\mu+\frac{d+3}{2})/2}} \\ \times (1-s)^{\mu}\,ds \\ \leqslant cn^{-\mu-3/2}\frac{1}{(1-\langle \mathbf{x},\mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}+n^{-2})^{(\mu+\frac{d+3}{2})/2}}.$$

Multiplying the estimate by $\mathcal{O}(1)\,n^{\mu+\frac{d+2}{2}}\sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}$ gives the third term in the stated estimate of $|A_n(\mathbf{x},\mathbf{y})|$. The proof is complete. ∎

PROPOSITION 3.5. *For $-1/2 < \mu < 0$ and $d \geqslant 3$,*

$$J_2(\mathbf{x}) := \int_{B^d} |A_n(\mathbf{x},\mathbf{y})|\,W_\mu(\mathbf{y})\,d\mathbf{y} \leqslant cn^{\frac{d-1}{2}}.$$

Proof. The estimate of $A_n(\mathbf{x},\mathbf{y})$ in the previous lemma shows that $J_2(\mathbf{x})$ is bounded by three terms. The first term is the integral of $n^{\mu+\frac{d}{2}}\,|P_n^{(\mu+\frac{d}{2},\,\mu+\frac{d-2}{2})}(\langle \mathbf{x},\mathbf{y}\rangle)|$ on B^d, which is the same as the bound of J_1 in Proposition 3.3. The third term is bounded by $cn^{\frac{d-1}{2}}$ times $D(\mathbf{x})$ and it suffices to prove the inequality

$$D(\mathbf{x}) := \int_{B^d}\frac{\sqrt{1-\|y\|^2}}{(1-\langle \mathbf{x},\mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}+n^{-2})^{(\mu+\frac{d+3}{2})/2}}\,W_\mu(\mathbf{y})\,d\mathbf{y} \leqslant c.$$

First we use (3.1) to get

$$D(\mathbf{x}) = \sigma_{d-2}\int_0^1\int_{-1}^1\frac{(1-s^2)^{\frac{d-3}{2}}\,ds\,r^{d-1}(1-r^2)^{\mu}\,dr}{(1-r\,\|\mathbf{x}\|\,s - \sqrt{1-r^2}\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{(\mu+\frac{d+3}{2})/2}}.$$

Then we split the outer integral into two parts, one over $[0, 4\,\|\mathbf{x}\|]$ and the other over $[4\,\|\mathbf{x}\|, 1]$; we denote the two terms by $D_{[0,4\,\|\mathbf{x}\|]}(\mathbf{x})$ and $D_{[4\,\|\mathbf{x}\|,1]}(\mathbf{x})$, respectively.

For the second part, $4\,\|\mathbf{x}\| \leqslant r$ or $\|\mathbf{x}\| \leqslant r/4$, which implies that

$$1-r\,\|\mathbf{x}\|\,s-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2} \geqslant 1-\frac{r^2}{4}-\sqrt{1-r^2}$$

$$=\frac{r^2}{1+\sqrt{1-r^2}}-\frac{r^2}{4} \geqslant \frac{r^2}{4}.$$

Thus, since $\frac{d-5}{2}-\mu \geqslant -1-\mu > -1$ for $d \geqslant 3$ and $\mu < 0$, we conclude that

$$D_{[4\,\|\mathbf{x}\|,1]}(\mathbf{x}) \leqslant c \int_{4\,\|\mathbf{x}\|}^{1} r^{\frac{d-5}{2}-\mu}(1-r^2)^{\mu}\,dr \leqslant c.$$

For the first part, $\|\mathbf{x}\| \geqslant r/4$; the fact that $1-r\,\|\mathbf{x}\|\,s-\sqrt{1-r^2}\sqrt{1-\|\mathbf{x}\|^2} \geqslant 0$ implies that

$$1-r\,\|\mathbf{x}\|\,s-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2} = 1-r\,\|\mathbf{x}\|-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+r\,\|\mathbf{x}\|\,(1-s)$$

$$\geqslant r\,\|\mathbf{x}\|\,(1-s) \geqslant \frac{r^2}{4}(1-s).$$

For $d \geqslant 4$, using $\frac{d-3}{2}-\frac{1}{2}\left(\mu+\frac{d+2}{2}\right)=\frac{d-4}{4}-1-\frac{\mu}{2} \geqslant -1-\frac{\mu}{2} > -1$ we conclude that

$$D_{[0,4\,\|\mathbf{x}\|]}(\mathbf{x}) \leqslant c \int_0^{4\,\|\mathbf{x}\|} \frac{r^{\frac{d-4}{2}-\mu}(1-r^2)^{\mu}\,dr}{(1-r\,\|\mathbf{x}\|+\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{1/4}}.$$

Changing variables $r=\cos\theta$ and $\|\mathbf{x}\|=\cos\phi$ and enlarging the integral domain, we obtain

$$D_{[0,4\,\|\mathbf{x}\|]}(\mathbf{x}) \leqslant c \int_0^{\pi/2} \frac{(\cos\theta)^{\frac{d-4}{2}-\mu}\,(\sin\theta)^{2\mu+1}}{\left(2\sin^2\dfrac{\theta-\phi}{2}+n^{-2}\right)^{1/4}}\,d\theta$$

$$\leqslant c \int_0^{\pi/2} \frac{(\pi/2-\theta)^{\frac{d-4}{2}-\mu}\,\theta^{2\mu+1}}{(|\theta-\phi|+n^{-1})^{1/2}}\,d\theta \leqslant c$$

for $d \geqslant 4$, since $\frac{d-4}{2}-\mu > 0$ for $\mu < 0$ and $2\mu+1 > 0$. In the case $d=3$, we have

$$D_{[0,4\,\|\mathbf{x}\|]}(\mathbf{x}) = \sigma_{d-2} \int_0^{4\,\|\mathbf{x}\|}$$

$$=\int_{-1}^{1} \frac{ds\,r^2(1-r^2)^{\mu}\,dr}{(1-r\,\|\mathbf{x}\|\,s-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{(\mu+3)/2}}$$

$$= \frac{2}{\mu+1} \int_0^{4\|\mathbf{x}\|} \frac{1}{r\,\|\mathbf{x}\|} \Bigg[\frac{1}{(1-r\,\|\mathbf{x}\|-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{\frac{\mu+1}{2}}} - \frac{1}{(1+r\,\|\mathbf{x}\|-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{\frac{\mu+1}{2}}} \Bigg] r^2(1-r^2)^\mu\,dr$$

$$\leqslant c \int_0^{4\|\mathbf{x}\|} \frac{1}{(1-r\,\|\mathbf{x}\|-\sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{\frac{\mu+1}{2}}} (1-r^2)^\mu\,dr$$

since $\|\mathbf{x}\| \geqslant r/4$. Again using $r = \cos\theta$ and $\|\mathbf{x}\| = \cos\phi$, we conclude that

$$D_{[0,\,4\,\|\mathbf{x}\|]}(\mathbf{x}) \leqslant c \int_0^{\pi/2} \frac{\theta^{2\mu+1}}{(|\theta-\phi|+n^{-1})^{\mu+1}}\,d\theta \leqslant c,$$

since $\mu+1<1$ for $\mu<0$ and $2\mu+1 \geqslant 0$. This completes the estimate of $D(\mathbf{x})$.

Finally we turn to the second term in the estimate of $A_n(\mathbf{x}, \mathbf{y})$ in the lemma. The integral of this term is bounded by $cn^{\frac{d-1}{2}}$ times $E(\mathbf{x})$ and we need to prove

$$E(\mathbf{x}) := \int_{B^d} \frac{1}{(1-\langle \mathbf{x}, \mathbf{y}\rangle - \sqrt{1-\|\mathbf{x}\|^2}\,\sqrt{1-\|\mathbf{y}\|^2}+n^{-2})^{(\mu+\frac{d+1}{2})/2}} W_\mu(\mathbf{y})\,d\mathbf{y} \leqslant c.$$

The proof is similar to that of $D(\mathbf{x})$. Again using (3.1) we get

$$E(\mathbf{x}) = \sigma_{d-2} \int_0^1 \int_{-1}^1 \frac{(1-s^2)^{\frac{d-3}{2}}\,ds\,r^{d-1}(1-r^2)^{\mu-\frac{1}{2}}\,dr}{(1-r\,\|\mathbf{x}\|\,s - \sqrt{1-r^2}\,\sqrt{1-\|\mathbf{x}\|^2}+n^{-2})^{(\mu+\frac{d+1}{2})/2}}.$$

Comparing with the formula for $D(\mathbf{x})$, we lost $(1-r^2)^{1/2}$ in the numerator but gain $1/2$ power in the denominator. We again split the outer integral into two parts, one over $[0, 4\,\|\mathbf{x}\|]$ and the other over $[4\,\|\mathbf{x}\|, 1]$. The second part, $E_{[4\,\|\mathbf{x}\|,\,1]}(\mathbf{x})$, estimated in the same way as $D_{[4\,\|\mathbf{x}\|,\,1]}(\mathbf{x})$, is bounded by

$$E_{[4\,\|\mathbf{x}\|,\,1]}(\mathbf{x}) \leqslant c \int_{4\,\|\mathbf{x}\|}^1 r^{\frac{d-3}{2}-\mu}(1-r^2)^{\mu-\frac{1}{2}}\,dr \leqslant c.$$

The first part, $E_{[0,\,4\,\|\mathbf{x}\|]}(\mathbf{x})$, is also estimated as $D_{[0,\,4\,\|\mathbf{x}\|]}(\mathbf{x})$, but much easier. Indeed, since $\frac{d-3}{2}-\frac{1}{2}(\mu+\frac{d+1}{2}) = \frac{d-3}{4}-1-\frac{\mu}{2} > -1$ for $d \geqslant 3$, it follows that

$$E_{[0,\,4\,\|\mathbf{x}\|]}(\mathbf{x}) \leqslant c \int_0^{4\,\|\mathbf{x}\|} r^{\frac{d-3}{2}-\mu}(1-r^2)^{\mu-\frac{1}{2}}\,dr \leqslant c.$$

This completes the estimate of $E(\mathbf{x})$ and the proposition. ∎

We now deal with the third term in the formula of $K_n(W_\mu)$ in Theorem 2.1. There are two terms, we only need to work with the first one,

$$F_n(\mathbf{x}, \mathbf{y}) := \int_0^1 K_n(w_{\mu+\frac{d-1}{2}}; 1, \langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, t) \times (1+t)^{\mu-2}(1-t)^{\mu}\, dt;$$

the other one is similar. We have

Proposition 3.6. *For $-1/2 < \mu < 0$ and $\mathbf{x} \in B^d$,*

$$J_3 := \int_{B^d} |F_n(\mathbf{x}, \mathbf{y})|\, W_\mu(\mathbf{y})\, d\mathbf{y} \leqslant c n^{\mu+\frac{d-1}{2}}.$$

Proof. Using (2.3) and Lemma 3.2, we get

$$\begin{aligned} |F_n(\mathbf{x}, \mathbf{y})| &\leqslant c n^{\mu+\frac{d}{2}} \int_0^1 |P_{n-1}^{(\mu+\frac{d}{2}, \mu+\frac{d-2}{2})}(\langle \mathbf{x}, \mathbf{y} \rangle + \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, s)| \\ &\quad \times (1+s)^{\mu-2}(1-s)^{\mu}\, ds \\ &\leqslant c n^{\mu+\frac{d-1}{2}} \int_0^1 \frac{(1+s)^{\mu-2}(1-s)^{\mu}\, ds}{(1-\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2}\, s + n^{-2})^{(\mu+\frac{d+1}{2})/2}} \\ &\leqslant c n^{\mu+\frac{d-1}{2}} \frac{1}{(1-\langle \mathbf{x}, \mathbf{y} \rangle - \sqrt{1-\|\mathbf{x}\|^2}\sqrt{1-\|\mathbf{y}\|^2} + n^{-2})^{(\mu+\frac{d+1}{2})/2}}. \end{aligned}$$

Apart from the power of n, the last term is bounded by the second term of $A_n(\mathbf{x}, \mathbf{y})$ in Lemma 3.4. Hence, the proof of Proposition 3.5 applies. ∎

Together, the estimates of J_1, J_2 and J_3 conclude the proof of Theorem 1.2.

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